Public good provision with many agents: the k-success technology

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Abstract

In this paper, we consider a class of public good provision problems in which the production function takes the form of k-success technology, an extension of the direct provision technology considered in Behringer (2013). These models are suitable to describe the free-rider problems in which there are a large number of agents who are both users and beneficiaries of a public good at the same time, e.g. open-source software or social networks. We provide results on asymptotic efficiency which connect a negative result of Mailath and Postlewaite (1990) and a positive result of Hellwig (2003), as well as a set of simple examples which allow us welfare comparison with the standard technologies.

1 Introduction

Investigations of the provision of public goods have a long tradition in the theory of public finance. Such public goods are known to raise the free-riding problem at least since Samuelson (1954), and it can easily be shown that even in full information context the problem becomes more severe when there are many agents involved.[1]

Accepting the fact that there usually is asymmetric information and the provider of the public good (often assumed to be the government) does not know in advance how people value the public good individually (and thus in aggregate), the optimal decision about provision (or sometimes about the optimal provision level) is also not known ex-ante. The literature of *mechanism design* has taken this information asymmetry very seriously and tried to offer support to policy makers regarding these issues by designing a device that intends to

[1] We are grateful for comments by Claude d'Aspremont, Ehud Kalai, Thomas Weber, and participants of the UECE Lisbon meeting 2014 and a seminar at Ecole Polytechnique Federale de Lausanne.

elicit these true individual preferences, that is to satisfy *incentive compatibility* constraints.[2]

Much of the early theoretical work has been conducted on the basis of the proposal made by Groves (1973), often called the Vickrey-Clarke-Groves mechanism which allows for implementation in *dominant strategies*, i.e. optimal own announcements about valuations will not depend on what other agents do or about expectations about their true valuations.

This form of implementation, also used in Groves and Ledyard (1977), proposed a solution to the free-rider problem. Important contributions in this area are summarized in the important and comprehensive work of Green and Laffont (1979), who also look at the benefits of large numbers for the possibility to *sample* preferences from only a subset of the population. If there is a cost to sampling, an optimal sample size can be calculated.

A rising concern of theorists for the need to advocate mechanisms to the policy maker that are neutral on the budget, or *balanced-budget*, i.e. where the incentive scheme underlying the mechanism does not require extra governmental funding, i.e. money that should rather go to the provision of the public good itself, has led them to look for a weaker form of implementation, namely *Bayesian implementation*. Here one's optimal strategy will depend on the own valuation and information about the other's type (usually the statistical properties of some parent distribution that is assumed to be commonly known). Hence the restriction this implies for the form that a mechanism may take are strict. The important proposal in this direction is by d'Aspremont and Gerard-Varet (1979) and by Arrow (1979), but in this mechanism another attractive property fails, namely agents may prefer to opt out of the mechanism altogether, i.e. Bayesian *individual rationality* is violated.[3]

Operating within the environment of Bayesian implementation, there have been two recent tendencies that have attracted particular interests. First, there has been an investigation of mechanisms for *more specific cases of public goods* with non-rival properties, e.g. Baliga and Maskin (2003) investigate mechanisms for the environment.

Secondly, there has been an interest in the performance of Bayesian mechanisms in *large economies,* i.e. what one may be able to say about the performance of a mechanism once the number of agents for which the public good is to be provided gets very large. Early fundamental results such as those by Mailath and Postlewaite (1990) have been predominantly negative. They conclude that provision of the public good converges to zero as the number of agents

[2] For a very recent introduction to these issues with much emphasis on the public good provision problem, see Börgers (2014)

[3] For a recent contribution proportionally wakening the individual rationality requirement see Rong (2014).

increases. The underlying intuition for the result is that, unlike in a private good setting where efficiency can usually be obtained in the limit (e.g. Cripps and Swinkels (2006)), with many agents the probability of being the agent who by his truthful announcement of valuation *tilts* the balance between non-provision and provision in favour of the latter is strictly decreasing. As this possibility of being pivotal for the outcome of the provision effort is commensurable with the amount that an individual is willing to voluntarily contribute for the good which comes at a constant per-capita cost (e.g. national defense), the former effect will be dominating with sufficiently many agents. This leads to non-provision in the limit.

For the *open-source software context,* this translates as follows: If there are sufficiently many agents, I may just wait for yet another module of the software (e.g. a diver for my own printer) to be developed, even if I may quite easily write this piece of source code myself. In other words, the probability of being pivotal decreases as there are many agents so that following the findings of Mailath and Postlewaite (1990) we should expect the development process to be slow when there is a myriad of agents involved.

Much more on the general questions and explanations for motivation of contributors that arise in this specific context can be found in the survey of Gandal and Fershtman (2011). The authors argue, based on the work of Johnson (2002), that in models of voluntary provision of public goods "one needs to assume that the primary motivation of developers is the "consumption" or the use of the final program." We show in this paper that this assumption can be relaxed and one may obtain voluntary provision of a public good such as OSS even when the consumption aspect (or even altruism) is not the major driving force of individual behavior.

Hellwig (2003) has also looked at the issue for both an a priori bounded and an unbounded public good, and finds that assuming a fixed cost for the public good (e.g. a lighthouse) is sufficient for the negative limit result of Mailath and Postlewaite (1990) to be reversed. More precisely, he is able to show that the second-best provision levels converge in distribution to the first-best, (full information) levels. For the unbounded case, he shows that the difference between the second-best and the first-best levels becomes large and hence there is serious underprovision of the public good with many agents. In the latter analysis, Hellwig provides a set of sufficient conditions on the cost function.

Recent work of Behringer (2013) contributes to both these tendencies. The work is motivated by a dissatisfaction with the sensitivity of the limit results with respect to the ex-ante assumption about the cost technology, and also by the observation of rapid development of *public information goods* such as Open-source Software (despite there being a myriad of agents involved). The author is able to endogenize the choice of the cost technology at the expense of having a specific provision technology. A major aim of this paper is to extend the provision technology to a more general one.

The *direct provision technology* has the feature that the agents who traditionally only consume the public good are the very same agents who may or may not provide it. There is thus no need for government provision as in the standard public good provision context. This form of provision seems to dominate important recent development in information technologies like the rapid developments of multi-lingual online encyclopaedia such as Wikipedia, or complex networks such as Myspace or Facebook.

In particular, the above mentioned case of Open-Source Software such as Linux (see Raymond (1996) for fascinating details of this movement and the homepage of the MIT Free/Open-Source Research Community containing virtually hundreds of articles that address these issue form multidisciplinary points of view) has intrigued economists for quite some time now. Some have even argued it to be some third way of production that efficiently aggregates many voluntary contributors in one production process. Others such as Lerner and Tirole (1994) argue that rather delayed rewards to specific programmers may be responsible for their willingness to contribute. The occupation with cases of such social production excites disciples well beyond the focus of economists i.e. Benkler (2002) for a legal approach.

2 The Model

There are n ex ante homogeneous agents who collectively exert efforts to provide an indivisible public good. For each agent $i \in \{1, \cdots, n\} =: N$, her valuation $\theta_i \in \Theta := [\underline{\theta}, \bar{\theta}]$ of the public good is private information, and is independently and identically distributed from the density function f, which we assume is continuous and strictly positive on Θ. Let r be the probability that the indivisible public good is provided. The preferences of the agents are assumed to be quasi-linear:

$$u_i = \theta_i r - t_i$$

where t_i is a payment of i.

A novelty of our model is that we consider explicitly the production technology as a function of individual effort levels. More precisely, we assume

$$r = \varphi_k (p_1, \cdots, p_n) \tag{1}$$

where $p_i \in [0, 1]$ is the effort level of agent i, which we later identify as the probability of individual success.

An example of such technology is called *direct provision technology* under which each agent i can provide the good with some probability p_i directly. In

Behringer (2013), the total probability of the good being provided independently by n agents is assumed to be

$$r = 1 - \prod_{i=1}^{n} (1 - p_i). \tag{2}$$

A key feature in this technology is that any agent can bring about the provision of the good on his own.

Let z_i be the net side payment of agent i. Assuming that the preferences are additively separable in effort, we have:

$$u_i = \theta_i r - L(p_i) - z_i$$

where function L represents the disutility from effort. Assume $L' > 0$.

Employing the revelation principle, we restrict ourselves to incentive compatible direct mechanisms. A mechanism is a pair of functions $\langle \mathbf{p}, \mathbf{z} \rangle$ where $\mathbf{p} = (p_i)_{i \in N}$ is the vector of effort allocations with $p_i : \Theta^n \to [0,1]$ for $\forall i$, and $\mathbf{z} = (z_i)_{i \in N}$ is the vector of net side payments with $z_i : \Theta^n \to \mathbb{R}$. Let $\boldsymbol{\theta} = (\theta_i)_{i \in N}$ be the vector of valuations. A mechanism is *feasible* if the side payments satisfy:

$$\sum_{i=1}^{n} z_i(\boldsymbol{\theta}) \geq 0 \text{ for } \forall \boldsymbol{\theta}. \tag{3}$$

2.1 A generalized technology

In this paper we generalize the direct technology to the *k-success technology* in which global success is obtained if and only if k or more individual successes are realized. Remind that $(p_i)_{i \in N}$ are independent probabilities of *individual successes*, and r is the probability of a *global success*. In some sense this generalized setup takes into account the cooperative nature of developments of the public goods under scrutiny. Assuming that individual successes are independent across agents, the k-success technology satisfies:

$$\varphi_k(p_1, \cdots, p_n) = \sum_{\{S \subset N \mid |S| \geq k\}} \prod_{i \in S} p_i \prod_{i \notin S} (1 - p_i). \tag{4}$$

Now, we assume that disutility from effort is a linear function and normalize that $L(p_i) = p_i$. Efficient production is derived by solving:

$$C(r) := \min_{p} \sum_{i} p_i \text{ subject to } \varphi_k(p_1, \cdots, p_n) \geq r. \tag{5}$$

We then obtain the first result.

Lemma 1 *Suppose $k = 1$. For any r, efficient production is obtained when there exists unique i such that $p_i = r$ and $p_j = 0$ $\forall j \neq i$. The cost function is $C(r) = r$.*

Proof. First, note that for any r, global success rate r is attained by assuming $p_i = r$ for some i and $p_j = 0$ for $\forall j \neq i$. Hence, $C(r) \leq r$. The solution of (5) is obtained by considering the following dual problem:

$$\max_p \varphi_k (p_1, \cdots, p_n) \text{ subject to } \sum_i p_i \leq c. \tag{6}$$

Since $C(r) \leq r \leq 1$, we only consider the case $c \leq 1$. Using (2), maximizing φ_1 is equivalent to minimizing $\prod_i \bar{p}_i$ where $\bar{p}_i = 1 - p_i$. Under the constraint that $\sum_i \bar{p}_i \geq n - c \geq n - 1$, the solution is a corner one: $\bar{p}_i < 1$ for some i and $\bar{p}_j = 1$ for $\forall j \neq i$. We thus obtain solutions: $p_i = r$ for some i and $p_j = 0$ $\forall j \neq i$. It then follows that $C(r) = r$. ■

Note that $C(r)$ is independent of n. We thus find that absent an implementation assumptions that requires all agents to participate in the provision effort symmetrically a delegation of the provision to one agent will lead to a efficient cost function that is *independent* of the number of agents. Hence the sufficient condition for a positive limit result as investigated in Hellwig (2003) is satisfied.

Now let us consider the k-success technology for $k \geq 2$. For any k and n, there exists an integer $K \in [k, n]$ such that the solution is interior (i.e. $p_i \in (0, 1)$) for exactly K members. Using the symmetric nature of the problem (5), the individual efforts should be equal for such K members. More precisely, consider the first order condition of the dual problem (6):

$$\frac{\partial \varphi_k}{p_i} = \frac{\partial \varphi_k}{p_j} \text{ for } j \neq i$$

which implies that the impact of individual efforts on the total provision (the global success) should be equal amongst agents, as long as the solution is interior. Then, the *inverse cost function* of the interior solution follows the binomial distribution in which k or more successes are obtained among K independent draws with probability c/K, i.e.:

$$r(K, k, c) = \sum_{i=k}^{K} \binom{K}{i} \left(\frac{c}{K}\right)^i \left(1 - \frac{c}{K}\right)^{K-i}. \tag{7}$$

The form of the inverse cost function determines the efficient allocation of individual efforts.

Lemma 2 *Suppose $k \geq 2$. The inverse cost function for the interior solution is increasing and s-shaped in c. There is a critical value c^* such that it is convex for $c < c^*$ and concave for $c > c^*$.*

Proof. First we show that the derivative of the inverse cost function is positive and single peaked.

$$\begin{aligned}
\frac{\partial r}{\partial c} &= \sum_{i=k}^{K}\binom{K}{i}\frac{i}{K}\left(\frac{c}{K}\right)^{i-1}\left(1-\frac{c}{K}\right)^{K-i}-\sum_{i=k}^{K}\binom{K}{i}\frac{K-i}{K}\left(\frac{c}{K}\right)^{i}\left(1-\frac{c}{K}\right)^{K-i-1} \\
&= \sum_{i=k}^{K}\binom{K-1}{i-1}\left(\frac{c}{K}\right)^{i-1}\left(1-\frac{c}{K}\right)^{K-i}-\sum_{i=k}^{K}\binom{K-1}{i}\left(\frac{c}{K}\right)^{i}\left(1-\frac{c}{K}\right)^{K-i-1} \\
&= \binom{K-1}{k-1}\left(\frac{c}{K}\right)^{k-1}\left(1-\frac{c}{K}\right)^{K-k} \qquad (8)
\end{aligned}$$

is a binomial function, and thus positive and single-peaked. The peak of this derivative and hence the *point of inflection* c^* of the inverse cost function can be calculated explicitly. Solving

$$\frac{\partial^2 r}{\partial c^2}=\binom{K-1}{k-1}\left(\frac{c}{K}\right)^{k-2}\left(1-\frac{c}{K}\right)^{K-k-1}\frac{1}{K}\left(k-1-\frac{K-1}{K}c\right)=0,$$

we have:

$$c^*=\frac{K(k-1)}{K-1}.$$

As the second order delivative is decreasing in c,

$$\underset{c<c^*}{sgn}\left(\frac{\partial^2 r}{\partial c^2}\right)>0 \text{ and } \underset{c<c^*}{sgn}\left(\frac{\partial^2 r}{\partial c^2}\right)<0,$$

and we have a rising inflection point with convexity for $c<c^*$ and concavity for $c>c^*$. ■

We thus know that the cost function $C(r)$ is increasing, first concave and then convex in r, as long as the solution of the dual problem is interior. We now investigate which interior solutions are obtained as a function of r.

Proposition 3 *Suppose $k \geq 2$. For small r, the most efficient production is obtained with $p_i = p$ $\forall i$ (i.e. equal contribution $K = n$). For large r, the most efficient production is obtained when $p_i = p$ for exactly k agents and $p_j = 0$ otherwise (i.e. specialization $K = k$).*

Proof. (A sketch) Considering the inverse cost function (7), it suffices to show that for any $K_1 > K_2 \geq k$, $\exists \hat{c}$ such that $r(K_1,k,c) > r(K_2,k,c)$ if $c<\hat{c}$ and $r(K_1,k,c) < r(K_2,k,c)$if $\hat{c}<c\leq K_2$ (we can exclude $c>K_2$, since $r(K_2,k,c)$ is well-defined only for $c \leq K_2$.) As we showed in (8), $\frac{\partial r}{\partial c}(K,k,c)$ is single-peaked with the peak at $K(k-1)/(K-1)$. Since $K/(K-1)$ is decreasing in K, the peak moves to the left as K increases. For large K, the support of the single-peaked function $\frac{\partial r}{\partial c}(K,k,c)$ becomes larger, and the peak becomes lower, which implies that $\frac{\partial r}{\partial c}(K_1,k,c)$ is larger than $\frac{\partial r}{\partial c}(K_2,k,c)$ for small c, and the inequality is reversed as c increases. This implies that $r(K_1, k, c) > r(K_2, k, c)$ for small c, and the inequality is reversed for large c. ■

We thus find that for small r, the cost of employing more agents is lower than having only a smaller number of agents exerting effort. This is because the reduction in individual cost by involving efforts by many agents outweighs the increasing return to scale induced by the concavity of the cost function for small r.

We also find:

Lemma 4 *Specialization ($K = k$) always leads to a concave cost function.*

Proof. Letting $K = k$ in (7), the inverse cost function simplifies to:

$$r = \sum_{i=k}^{k} \binom{k}{i} \left(\frac{c}{k}\right)^i \left(1 - \frac{c}{k}\right)^{k-i} = \left(\frac{c}{k}\right)^k.$$

Hence we find that the cost function $C(r) = kr^{1/k}$ is *concave* in r for $\forall k > 1$. ■

As r increases, the optimal number of contributors K decreases from n to k. The global cost function $C_{n,k}(r)$ for fixed n and k is then determined as a collection of cost functions (by choosing the optimal K for each r), each of which is piecewisely defined as the inverse of $r(K, k, c)$ for $K = k, k+1, \cdots, n$.

2.2 The mechanisms

Once we specify the effort allocations $\mathbf{p} = (p_i)_{i \in N}$ with $p_i : \Theta^n \to [0, 1]$ for $\forall i$, the probability of public good provision is uniquely determined by the production function (1). With a slight abuse of notation, denote such function as $r : \Theta^n \to [0, 1]$.

Define the interim provision probability function as:

$$\rho_i(\theta_i) \equiv \int r(\theta_i, \boldsymbol{\theta}_{-i}) dF^{n-1}(\boldsymbol{\theta}_{-i}). \tag{9}$$

Denote the cost function as $C_n(r(\boldsymbol{\theta}))$ derived from the inverse cost functions above. Aggregating, amalgamating, and adequately transforming the individual constraints on the mechanism, for the total contract then the following well-known result is obtained.

Lemma 5 *For any probability of provision $r(\boldsymbol{\theta})$ such that $\rho_i(\theta_i)$ is non-decreasing in θ_i, there exist net side payments $\mathbf{z}(\boldsymbol{\theta})$ such that $\langle r(\boldsymbol{\theta}), \mathbf{z}(\boldsymbol{\theta}) \rangle$ satisfies interim incentive compatibility, interim individual rationality, and weak feasibility iff*

$$\int \left(\sum_{i=1}^{n} \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\boldsymbol{\theta}) - C_n(r(\boldsymbol{\theta})) \right) dF^n(\boldsymbol{\theta}) \geq 0. \tag{10}$$

Proof: See Behringer (2013).■

It has been shown that the step from dominant strategy implementation and the implied non-balancedness to Bayesian implementation with its requirement of Bayesian balanced-budget condition (or weak feasibility here) is not too bothersome even if, as a policy maker, we should be concerned with ex-post balanced-budget condition. This is because once we have found a mechanism that indeed satisfies incentive compatibility, individual rationality and balanced-budget condition at the individually expected level an alternative mechanism can be constructed that satisfies ex post balanced-budget condition (see Börgers and Norman, 2008).

The mechanism designer solves the following program for the weakly feasible overall mechanism that implements the contracts under interim individual rationality and incentive compatibility and generates non-negative expected social benefit. This program is: Maximize expected social welfare programme $\mathcal{P}$:

$$\max_{r(\boldsymbol{\theta})} \left\{ \int \left(\sum_{i=1}^{n} \theta_i r(\boldsymbol{\theta}) - C_n(r(\boldsymbol{\theta})) \right) dF^n(\boldsymbol{\theta}), 0 \right\} \tag{11}$$

$$\text{s.t.} \quad \rho_i(\theta_i) \text{ non-decreasing in } \theta_i, \text{ and}$$

$$\int \left(\sum_{i=1}^{n} \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\boldsymbol{\theta}) - C_n(r(\boldsymbol{\theta})) \right) dF^n(\boldsymbol{\theta}) \geq 0.$$

i.e. subject to *monotonicity*, which can be guaranteed by assuming that the distribution satisfies the *hazard rate condition*, i.e. that the virtual utility $\theta_i - (1 - F(\theta_i))/f(\theta_i)$ is non-decreasing in θ_i.

The *ex-post efficient* (or the **first-best**) provision rule given by the first order condition for (11) equates marginal cost and marginal benefit (here equal to total benefit as we are concerned with a good that is non-rival in consumption) and thus:

a) for the direct technology of Behringer (2013),

$$\frac{\partial C_n(r(\boldsymbol{\theta}))}{\partial r} = (1 - r)^{\frac{1}{n} - 1} = \sum_{i=1}^{n} \theta_i,$$

b) for the case of *specialization* of k agents,

$$\frac{\partial C_n(r(\boldsymbol{\theta}))}{\partial r} = r^{\frac{1-k}{k}} = \sum_{i=1}^{n} \theta_i,$$

c) for the general *k-success technology* case,

$$\frac{\partial C_n(r(\boldsymbol{\theta}))}{\partial r} = \frac{\partial C_{n,k}(r)}{\partial r} = \sum_{i=1}^{n} \theta_i$$

where the marginal cost, as we argued, is always positive but u-shaped, i.e. first decreasing and after $r(n,k,c^*)$ increasing again.

The first-best provision rule is thus
a)

$$r_a^{FB*}(\boldsymbol{\theta}) = 1 - \left(\sum_{i=1}^{n} \theta_i\right)^{-\frac{n}{n-1}}, \tag{12}$$

or b)

$$r_b^{FB*}(\boldsymbol{\theta}) = \left(\sum_{i=1}^{n} \theta_i\right)^{\frac{k}{1-k}},$$

or c)

$$r_c^{FB*}(\boldsymbol{\theta}) = r\left(K, k, \sum_{i=1}^{n} \theta_i\right).$$

This program, using a Lagrange multiplier approach programme $\mathcal{P}$ can be rearranged as:

$$\max_{r(\boldsymbol{\theta}),\lambda} \left\{ \begin{array}{c} (1+\lambda)\left(\int \left(\sum_{i=1}^{n} \theta_i r(\boldsymbol{\theta}) - k_n(r(\boldsymbol{\theta}))\right) dF^n(\boldsymbol{\theta})\right) \\ -\lambda \int \left(\sum_{i=1}^{n} \left(\frac{1-F(\theta_i)}{f(\theta_i)}\right) r(\boldsymbol{\theta})\right) dF^n(\boldsymbol{\theta}), 0 \end{array} \right\} \tag{13}$$

so that the **second-best** provision rule can be written as (see Lemma 4 in Behringer, 2013)

$$r_{\cdot}^{SB*}(\boldsymbol{\theta}) = \begin{cases} r_{\cdot}^{FB*}(\boldsymbol{\theta}) & \begin{array}{l} \text{if } (1+\lambda)\left(\sum_{i=1}^{n} \theta_i r_{\cdot}^{FB*}(\boldsymbol{\theta}) - k_n\left(r\left(\boldsymbol{\theta}\right)\right)\right) \\ > \lambda \sum \frac{1-F(\theta_i)}{f(\theta_i)} r_{\cdot}^{FB*}\left(\boldsymbol{\theta}\right) \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

2.3 Asymptotic efficiency

We are interested in asymptotic efficiency of the public good provision problem as the number of agents becomes large. Mailath and Postlewaite (1990) provide a sufficient condition for a negative result while a sufficient condition for asymptotic efficiency is provided in Hellwig (2003). In both papers conditions are given in terms of the cost function. In this paper, we consider a class of public good provision problems under k-success technologies, and provide both negative and positive results, filling the gap between the two extreme results.

In order to describe asymptotic properties, we consider a sequence of public good provision problems with k-success technology. More precisely, let $\{k_n\}_{n=1}^{\infty}$ be a sequence in which, for each $n \in \mathbb{N}$, a global success for the public provision is obtained if k_n or more individual successes are obtained.

Proposition 6 (A positive result) *Suppose that* $\lim_{n\to\infty}(k_n/n) = 0$*. Then the probability of public good provision converges to one as* $n \to \infty$*.*

Proof. Following Hellwig (2003), define the following correspondence:

$$\Psi(A) := \arg\max_r \{Au(r) - C_n(r)\}.$$

Then, we have $\lim_{A\to\infty}\Psi(A) = \{1\}$. In order to obtain the possibility result, all the arguments in the proof of Proposition 3 in Hellwig (2003) go through, except that we need to verify $C_n(1)/n \to 0$. Using Proposition 3, efficient production for large r is achieved when $p_i = p$ for exactly k_n members and $p_i = 0$ for the rest. Hence, $C_n(1) = k_n$. Then, $\lim_{n\to\infty} C_n(1)/n = k_n/n = 0$ by assumption. ■

Proposition 7 (A negative result) *Suppose that* $\lim_{n\to\infty}(k_n/n) = \alpha > 0$*. Then the probability of public good provision converges to zero as* $n \to \infty$*.*

Proof. As above, efficient production is achieved when $p_i = p$ for exactly k_n members and $p_i = 0$ for the rest. Hence, we have $C_n(1) = k_n$. Since $\lim_{n\to\infty}(k_n/n) = \alpha > 0$, for sufficiently large n, we have $C_n(1)/n > \alpha/2 > 0$. Therefore, $C_n(1) > \alpha n/2$ for large n. The sufficient condition (iv) of Theorem 2 in Mailath and Postlewaite (1990) is satisfied, implying that we have a negative result. ■

2.4 Limit results

Looking at the *limit behavior* one finds that: for a),

$$C_n(r(\boldsymbol{\theta}))_a = n(1-(1-r)^{\frac{1}{n}})$$

with

$$\lim_{n\to\infty} C_n(r(\boldsymbol{\theta}))_a = -\ln(1-r) < \infty \text{ for } r < 1$$

see Behringer (2013) Lemma 7.

For b), under specialization,

$$C_n(r(\boldsymbol{\theta}))_b = k\sqrt[k]{r}$$

and

$$\lim_{n\to\infty} C_n(r(\boldsymbol{\theta}))_b = k\sqrt[k]{r} < \infty$$

unless k is proportional to n.

For c), remember that the *k-success technology* is represented by the inverse cost function (7). Now in the case that n grows large, the probability of i successes converges to the *Poisson distribution*:

$$\lim_{n\to\infty}\binom{n}{i}\left(\frac{c}{n}\right)^i\left(1-\frac{c}{n}\right)^{n-i} = e^{-c}\frac{c^i}{i!}$$

with mean and variance c. Thus, the inverse cost function for the limit is

$$r_c^\infty = e^{-c} \sum_{i=k}^{\infty} \frac{c^i}{i!}$$

so that the first best limit solution equalizes marginal cost and marginal/total benefit, and therefore satisfies

$$\frac{\partial C_\infty(r(\boldsymbol{\theta}))}{\partial r_c} = \frac{(k-1)!}{e^{-C} C^{k-1}} = \sum_{i=1}^{\infty} \theta_i.$$

Hence it has been shown that specific technology with $k = 1$ as employed in Behringer (2013) generalizes to any $k > 1$ and that this generalized technology is equally sufficient to obtain a positive limit result (unless the amount of required successes turns out to be proportional to n which is unlikely for the examples envisaged above). As an important additional result we have shown that for a smaller desired total provision probability r, equal effort by all user/developers will be superior to specialization in cost and thus in welfare terms rather than being assumed.

Instead of looking at large sample properties in more detail, we will now investigate very simple examples for the specialization effect in the *k-success* technology. In order to compare how the mechanisms perform relative to other more standard mechanisms in small samples.

3 Examples

3.1 A specialization example

Let $n = 4$ and $k = 2$, i.e. the k-success technology requires at least two successes in 4 draws, i.e. with 4 agents exerting normed effort p.

Using the inverse cost function (7), in the case of *equal effort* for all 4 agents, we have

$$r_4 = \left(\frac{c}{4}\right)^4 + 4\left(\frac{c}{4}\right)^3\left(1 - \frac{c}{4}\right) + 6\left(\frac{c}{4}\right)^2\left(1 - \frac{c}{4}\right)^2.$$

If we let 3 agents specialize ($K = 3$), we have:

$$r_3 = \left(\frac{c}{3}\right)^3 + 3\left(\frac{c}{3}\right)^2\left(1 - \frac{c}{3}\right),$$

and if only 2 agents specialize ($K = 2$), we have:

$$r_2 = \left(\frac{c}{2}\right)^2.$$

We know from the above proposition that the inverse cost functions always intersect. Numerically we find that $r_4 = r_3$ at $c = 1.093$ and that $r_3 = r_2$ at

$c = 1.125$. Thus whereas it is equal sharing that cost dominates all others for $c < 1.093$ and full specialization dominates for $c > 1.125$, the range in between indicates the that also partial specialization (of *any* intermediate number of agents K with $k \leq K \leq n$) may be optimal. This follows from the continuous behavior of the inverse cost function for changes in n and k which implies that also their intersections will change continuously.

Hence again we show that full specialization of exactly k agents will be optimal if the total required success probability is high. Inversely if the total requited success probability requirement is low, i.e. r is low, then it is most cost effective to develop with all agents putting in effort. All this is in line with the above Proposition.

3.2 Small-sample, uniform examples

The following gives explicit calculations for the smallest possible sample, comparing the *k-success technology* for $k = 1$ and the technologies of Mailath and Postlewaite (1990) and Hellwig (2003).

For a special case with $n = 2$ agents, suppose that each θ_i is uniformly, identically and independently distributed on the unit interval $[0, 1]$.

Lemma 8 *Assuming equal contributions in the direct technology and $k = 1$. The utilitarian designer chooses a mechanism with the allocation rule*

$$r^* = \begin{cases} 1 - (\theta_1 + \theta_2)^{-2} & \text{if } \theta_1 + \theta_2 \gtrsim s^+ \\ 0 & \text{otherwise} \end{cases}$$

where $s^+ \approx 1.4$.

Proof: See Appendix.■

The first-best allocation implies production if $\theta_1 + \theta_2 > 1$, and th second-best only leads to production if $\theta_1 + \theta_2 > 1.4$, so that the second best is bounded from the first best.

For the standard technologies we find the following result:

Lemma 9 *With a standard technology as in Mailath and Postlewaite (1990) or Hellwig (2003), $c = \frac{1}{2}n$ or $c = 1$, the utilitarian designer chooses a mechanism with the allocation rule*

$$r^* = \begin{cases} 1 & \text{if } \theta_1 + \theta_2 \gtrsim s^{++} \\ 0 & \text{otherwise} \end{cases}$$

where $s^{++} \approx 1.25$.

Proof: See Börgers (2014) p.59.■

We see that for this particular example with $n = 2$ and $k = 1$, the standard technologies are welfare superior.

4 Conclusion

The above analysis and the work in Behringer (2013) have shown that for voluntary provision of public goods one does not need to assume that the primary motivation of (Open-Source Software) developers is the consumption use or the use of the final program or even straightforward altruism.

This finding clearly does not claim exclusivity and the plethora of alternative theoretical and empirical investigations remain relevant and useful. On the other hand it shows the traditional framework of mechanism design theory and its classical motivational assumptions are flexible enough the encompass more recent development and that previous negative results can be overturned when the supply side of public good provision is taken seriously.

From the above examples for the performance of different standard Provision Technologies and that of the direct Provision Technology as in Behringer (2013) we can conclude that despite the latter's attractive properties in *large* samples it may perform worse in *small* ones.

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6 Appendix

Proof of Lemma 8. Total costs reduce to

$$c(r(\boldsymbol{\theta})) = 2 - 2\sqrt{(1 - r(\boldsymbol{\theta}))} \in [0, 2]$$

which is a *strictly convex function* over the domain of $r(\boldsymbol{\theta}) \in [0, 1)$. The first best provision rule is

$$r^{FB*}(\boldsymbol{\theta}) = \begin{cases} 1 - (\theta_1 + \theta_2)^{-2} & \text{if } \theta_1 + \theta_2 > 1 \\ 0 & \text{otherwise} \end{cases}.$$

The second best provision rule reduces to

$$\begin{aligned} r^{SB*} &= (1 - (\theta_1 + \theta_2)^{-2}) \text{ if } \theta_1 + \theta_2 > \\ & 2\frac{(\theta_1 + \theta_2) - (\theta_1 + \theta_2)^2}{1 - (\theta_1 + \theta_2)^2} + \frac{\lambda}{1 + \lambda}(2 - \theta_1 - \theta_2) \end{aligned}$$

The quadratic on the RHS of the inequality constraint has two roots

$$\theta_1 + \theta_2 = \frac{1}{2(1 + 2\lambda)}\left(1 + 2\lambda \pm \sqrt{(20\lambda^2 + 12\lambda + 1)}\right)$$

and so the inequality will hold if $\theta_1 + \theta_2$ is either "large" or "small". We take the larger root as for any $\lambda > 0$ the lower root is in the interval $\left(\frac{1}{2} - \frac{1}{2}\sqrt{5}, 0\right) = (-0.618\,, 0)$ which cannot be the case as $\theta_1 + \theta_2 \geq 0$, and the larger root is in the interval $\left(1, \frac{1}{2} + \frac{1}{2}\sqrt{5}\right) = (1, 1.618)$. We find that

$$r^{SB*} = \begin{cases} (1 - (\theta_1 + \theta_2)^{-2}) \text{ if } \theta_1 + \theta_2 > \underbrace{\frac{1}{2(1 + 2\lambda)}\left(2\lambda + 1 + \sqrt{(20\lambda^2 + 12\lambda + 1)}\right)}_{s} \\ 0 \text{ otherwise} \end{cases}$$

holds. The RHS of the condition is denoted by s and we seek to find the optimal

value for s. What is the bound on s? For $\lambda = 0$ we find $s = 1$ the first best outcome and

$$\lim_{\lambda \to \infty}(s) = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618.$$

Also s is monotone increasing in λ and hence we need only be concerned with $s \in [1, \frac{1}{2}(1 + \sqrt{5})]$, i.e. with the case $s > 1$ here. From above, the second-best costs are

$$c(r^{SB*}(\boldsymbol{\theta})) = 2 - 2\sqrt{(1 - r^{SB*}(\boldsymbol{\theta}))} = 2\frac{\theta_1 + \theta_2 - 1}{\theta_1 + \theta_2}.$$

As $s > 1$, a single agent cannot have a valuation that satisfies $\theta_1 + \theta_2 > s$. Hence,

$$\Pr\{\theta_1 + \theta_2 > s | \, s > 1\} = \Pr\{\theta_1 + \theta_2 > s | \, \theta_1 > s - 1, \theta_2 > s - \theta_1\}.$$

Expected costs are

$$C(s) = \int_{s-1}^{1} \int_{s-\theta_1}^{1} \left(2\frac{\theta_1 + \theta_2 - 1}{\theta_1 + \theta_2} \right) d\theta_2 d\theta_1 = 8 - 4\ln 2 - 6s + 4\ln s + s^2$$

so that they are strictly decreasing in the relevant range for s. Expected payments of one agent are

$$\int_{s-1}^{1} \int_{s-\theta_1}^{1} ((1 - (\theta_1 + \theta_2)^{-2})(2\theta_1 - 1))d\theta_2 d\theta_1 =$$
$$-\frac{1}{6}\frac{-2s + 18\,(\ln 2)\,s - 6s^2 + 9s^3 - 2s^4 - 18\,(\ln s)\,s - 12}{s}.$$

Total expected payments are twice this, so total expected payments are also strictly decreasing in s, given that *both* agents have to provide and hence a higher λ implies a lower total expected payment as expected. As a necessary condition for an optimum s is that it must be such that ex ante expected expenditure equals ex ante expected costs. The difference function (twice the expected payments minus cost) has a critical root $D(s^+ \approx 1.4) = 0$. ■