Network Effects, Spillovers, and Market Structure

Stefan Behringer Universität Bonn

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Abstract

This paper investigates the effect of spillovers in a model of endogenous technical change resulting from network effects e.g. in the market for hand-held video consoles on the existence of a lower bound to market concentration. (Keywords: Market Structure; New Goods)

1 Introduction

The traditional literature on market structure purports a negative relationship between market size and seller concentration on the grounds that for a given level of 'barriers to entry' an increase in market size should increase profitability of incumbents and thus lead to new entrants. This usually lowers the concentration ratio depending on prior skewness of the size distribution and on what share of a growing market the entrants can capture.

With *homogenous products* and competition *à la Cournot* it can be shown that the equilibrium price strictly decreases in the number of firms and approaches the competitive level in the limit (Walrasian equilibrium price) where price distortions will be minimized. The equilibrium number of firms that enter the industry will grow unbounded as market size increases. This implies that in large markets the concentration ratio declines to zero. Given

[0]Universität Bonn, Adenaueralle 24-42, 53113 Bonn. Tel +49 228 739208.
E-mail: s.behringer@gmx.de
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that firms compete *à la Bertrand* the presence of completely homogeneous products will lead to the 'Bertrand paradox'. With positive setup costs the industry will be a monopoly for *any* market size.

The uncomfortable side-by-side of Cournot and Bertrand models indicates the need for a more refined modelling approach in order to capture equilibrium entry behaviour of firms and the implied development of the concentration ratio in large markets.

A necessary and sufficient condition for non-fragmentation in large markets exists: There may be an upper limit on the total number of firms that can profitably enter the industry *independently* of market size. In order to investigate such an upper bound on the number of firms, Sutton (1998) develops a model of vertically differentiated products with *network effects*[1] which we will apply to the early video game industry of the 1990.

2 The Model

We extend Sutton's (1998) model by introducing spillovers. In the first period firms decide whether to enter at some cost. In the second how much of the good to supply to the market which clears at some equilibrium price p^*. In the third period again supply decisions are taken and the market is cleared. The strategic interdependence is à la Cournot.

We analyze the model in a stage game framework with each of the three periods corresponding to a stage. Firms are strategic players and its total payoffs correspond to the sum of second and third stage payoffs. The solution concept is Subgame Perfect Nash equilibrium (SPNE). Most proofs are relegated to an online Appendix.

The consumer's utility function is Cobb-Douglas of the form

$$U = (\sum_j u_j x_j)^{\delta} z^{1-\delta} \tag{1}$$

[1] See also its antecendent in Shaked & Sutton (1983), Sutton's (1991) investigation of advertising intensive industries and the role of spillovers in the cost-reduction model in Fudenberg & Tirole (1983).

where $x_j \in R_+$ is a 'quality good' produced by firm j (with quality parameter $u_j \in U$ with $U = \{u \,| u \in R_+ \text{ and } u \geq 1\}$), and $z \in R_+$ is a composite commodity.

Suppose that at the end of stage one N firms have entered ($N \in N_+$). They are indexed by $j = 1...N$ and are assumed to produce one quality good in each of periods two and three. Given a price vector $\mathbf{p} = (p_1, ...p_N)'$, $(p_j) \in R_+$ and a quality vector $\mathbf{u} = (u_1, ...u_N)'$ it is well known that a consumer's demand for the product of some firm $k \neq j$ takes the simple form

$$x_k = \begin{cases} 0 \text{ if } \dfrac{p_k}{u_k} > \min\limits_{j \neq k} \dfrac{p_j}{u_j} \end{cases} \tag{2}$$

Thus all quality goods with positive sales in equilibrium must have prices proportional to their qualities

$$p_j = \lambda u_j \;\forall\; j = 1, ...i, ...N. \tag{3}$$

Total expenditure on all quality goods of all consumers equals a constant fraction δ, of consumer income Y with aggregate expenditure $\delta Y \equiv S \in R_+$. The proportionality factor λ can be found from the aggregate budget constraint. Total consumer expenditure on all varieties of the quality good must equal total market size S so that

$$\sum_j p_j x_j = \sum_j \lambda u_j x_j = S \tag{4}$$

Proposition 1 *Under the previous assumptions, for any quality vector* $\mathbf{u} = (u_1, ...u_N)'$ *the stage game in period three has a unique Nash equilibrium in which it is firm* i'*s equilibrium strategy to produce output*

$$x_i^* = \frac{S}{c} \frac{N-1}{u_i \sum_{j=1}^{N} \frac{1}{u_j}} \left\{ 1 - \frac{N-1}{u_i \sum_{j=1}^{N} \frac{1}{u_j}} \right\} \tag{5}$$

which implies equilibrium prices of

$$p_i^* = \frac{c u_i}{N-1} \sum_{j=1}^{N} \frac{1}{u_j} \tag{6}$$

and equilibrium net profits given a common marginal cost are

$$S\pi(u_i\,|\mathbf{u}_{-i}) = S\left\{1 - \frac{N-1}{u_i \sum_{j=1}^{N} \frac{1}{u_j}}\right\}^2 \tag{7}$$

Proof: See Sutton, (1998), Appendix 15.1 or online Appendix.■

2.1 Network effects and spillovers: The video game industry

The following analysis is motivated by the market for hand-held consoles for video games. In the 1990s these devices were *new goods* the key competitors being Nintendo with the Game Boy and Sega's Game Gear. The history and structure of this "two-sided market" and their *important network effects*[2] imply a pricing strategy that was deemed to be "discriminatory" and "against the public interest" in the UK and is thus well documented (see Monopolies and Mergers Commission, 1995).

As noted by the Commission (1995, p.8) the market for video games and consoles had expanded extremely rapidly, the typical player being a 9-16 year old boy. Hand-held consoles were first launched in Japan and the US and came to Europe in 1989 where they were an instant success. (1995, p.65) The spread of information about the new devices took two important channels:

A) By "word of mouth" by users of the very same device (via specialist magazines, borrowing and swapping games) which we call an *intra-firm network effect*. As a console producer increases its output and sells to more individuals it becomes more likely, as the word spreads, that the firm will be able to sell its product to even more individuals in the next period.

B) The market of hand-held consoles was completely new in the early 1990. As advantages of hand-held consoles were still unknown to the majority of its potential customers, the purchase of *any* brand made it more likely that

[2]For a recent analysis of the video game industry with an empasis on network effects see Shankar & Bayus (2003). For an overview of the theory of "two-sided markets" with the video game industry as an example see Rochet & Tirole (2003) and Evans (2003). I am grateful to John Sutton for suggesting this example.

any variety of the product was consumed in the next period. Hence by selling more Game Boys and thereby making the attractions of hand-held consoles known to more potential consumers, Nintendo was also increasing demand for the Game Gear hand-held console made by Sega and vice versa. This effect we call *inter-firm network spillover.*

Thus the perceived quality level u_i of firm i is parameterized as

$$u_i = \max(1, \alpha x_i + \boldsymbol{\beta}'\mathbf{x}_{-i})^{\frac{1}{\gamma}} \ \forall \ i = 1...N \tag{8}$$

where γ is an elasticity parameter with $\gamma \to \infty$ implying no overall network effect. α is a scalar that determines the degree of intra-firm network effect with $\alpha \in [0, 1]$ and $\boldsymbol{\beta}'$ is a the transpose of a $(N-1) \times 1$ column vector with generic elements $(\beta_i) \in [0, 1]$ that gives the degree of inter-firm network spillovers. The quality level in stage two is assumed to be unity.

Equilibrium *stage three net profits of a deviant firm*, given all rival firms have the same quality level (so that $\mathbf{u}_{-i}$ has generic elements $(u_{-i}) = \bar{u}$) depend only on *relative* quality which can be seen by rewriting (7) as

$$S\pi(u_i \,|\mathbf{u}_{-i}) = S\left\{1 - \frac{1}{\frac{1}{N-1} + \frac{u_i}{\bar{u}}}\right\}^2 \tag{9}$$

As stage three profits in equilibrium depend only on the relative quality levels which in turn depend on the quantity choices in stage two via the learning technology we can calculate the optimal quantity choice as a subgame perfect equilibrium for stage two of the model.

Proposition 2 *Assuming full spillover symmetry ($\beta_i = \beta \ \forall \ i$) and $N > 1$, the subgame for periods two and three with N active firms has a unique symmetric subgame perfect equilibrium. The equilibrium involves period two quantity choices of*

$$x^* = \frac{S}{Nc}(1 - \frac{1}{N}) + \frac{1}{\gamma}\frac{\alpha - \beta}{\alpha + \beta(N-1)}2\left[N + \frac{1}{N} - 2\right]\frac{S}{N^2c} \tag{10}$$

Proof: See online Appendix.■

Lemma 1 The following comparative statics results hold for $N > 1$

$$\frac{\partial x^*}{\partial \beta} < 0, \frac{\partial x^*}{\partial \alpha} > 0, \frac{\partial x^*}{\partial \gamma} |_{\alpha > \beta} < 0, \frac{\partial x^*}{\partial \gamma} |_{\beta > \alpha} > 0$$

Proof: See online Appendix.■

The presence of network effects introduce additional strategic considerations into firms' profit maximizing output decisions. The more network effects can be used advantageously by the firm's competitors relative to the benefits they imply for the firm itself, the more its incentives to make use of such effects are distorted.

A larger overall network elasticity parameter, given intra-firm network effects dominates inter-firm network spillovers will decrease the subgame perfect equilibrium output in stage two and a firm will decide to incur less costs to benefit from it.

Proposition 3 *Equilibrium profits for firm i in the subgame perfect equilibrium of the above Proposition are*

$$\Pi_i = 2\frac{S}{N^2}\left\{1 - \frac{1}{\gamma}\frac{\alpha - \beta}{\alpha + \beta(N-1)}\left[N + \frac{1}{N} - 2\right]\right\} \tag{11}$$

Proof: See online Appendix.■

2.2 The Equilibrium amount of entry

Using backward induction we determine entry behaviour in stage one.

Proposition 4 *Given entry costs of $\varepsilon = 1$ and for all $\alpha, \beta \in [0, 1]$ there exists a unique subgame perfect equilibrium of the overall game where the equilibrium number of active firms N^* is given implicitly by the* largest integer *number N that satisfies*

$$2\left\{1 - \frac{1}{\gamma}\frac{\alpha - \beta}{\alpha + \beta(N-1)}\left[N + \frac{1}{N} - 2\right]\right\} \geq \frac{N^2}{S} \tag{12}$$

Proof: See online Appendix.■

The equilibrium number of firms N^* has comparative statics symmetric to those on profits and hence inverse to those on quantities: Larger inter-firm network spillovers and lower intra-firm network effects lead to larger output and lower equilibrium profits and therefore lower entry in stage one of the game. The effect of the overall elasticity of the network effect on the number of firms depends again on which effect dominates. If the own firm network effect is dominant, a lower overall network effect implies lower individual output and by backward induction a larger number of entrants and vice versa.

2.3 Finding a lower bound to concentration

Theorem 1 *If $\alpha > \beta(\gamma + 1)$ a lower bound to concentration exists. If $\alpha \leq \beta(\gamma + 1)$ the equilibrium number of firms N^* grows without bounds as the size of the market S gets large, therefore a lower bound to market concentration fails to exist.*

Proof: See online Appendix.■

For $\alpha < \beta$ a lower bound to concentration never exists (see online Appendix). If the spillover is large enough a firm will not find a *profitable deviation* and expand output beyond the standard Cournot quantity (in that sense a "high β reduces the impact of α") and as only relative qualities matter in equilibrium for large enough β it will refrain from doing so. Here individual network effects are too weak relative to spillover effects for *any* degree of the overall network effect γ. Thus we require individual intra-firm network effects to be stronger in order to get an "overproduction" result in stage two that lowers overall profits and bounds entry in stage one of the game. The asymmetry thus has to be in favour of intra-firm network effects.

However simple dominance of the form $\alpha > \beta$ is only necessary, not sufficient for the existence of a bound. Individual intra-firm network effects have to be supported by a strong overall network effect in order to be able to induce the overproduction that limits entry and creates a lower bound to concentration, i.e. satisfies the condition $\alpha > \beta(\gamma + 1)$.

Let us now compare these results to the results of the *standard Cournot model* repeated twice. Given that individual output levels are at the exact

Cournot level, i.e. if network effects are fully symmetric ($\alpha = \beta$) or nonexistent ($\gamma \to \infty$) (10) becomes $x^c = S/(Nc)(1 - \frac{1}{N})$. From the large literature on Cournot convergence it is well known that a lower bound to concentration fails to exist. For $\alpha \gtreqless \beta$ we have $x^c \lesseqgtr x^*$ if $\gamma < \infty$. From our limit analysis in Theorem 1 we know that for $\alpha \leq \beta(\gamma + 1)$ a lower bound fails to exits.

Whereas symmetric network effects are thus sufficient for a lower bound to concentration to fail they are not necessary as there is a range of $\beta \in [a/(\gamma + 1), \alpha]$ where individual output is above the Cournot level (and hence individual profits are lower) but there is still sufficient entry for the lower bound to concentration to fail in large markets. Clearly as the overall network effect is lower, i.e. $\gamma \to \infty$, this case is even more common.

Solving the left hand side (LHS) of the equilibrium condition for N

$$2\left\{1 - \frac{1}{\gamma}\frac{\alpha - \beta}{\alpha + \beta(N-1)}\left[N + \frac{1}{N} - 2\right]\right\} \geq 0 \tag{13}$$

we define $\tilde{N}(\alpha, \beta, \gamma)$ as the largest integer that satisfies (13), i.e. gives an upper bound on the equilibrium number of firms in the industry which implies a lower bound to market concentration in large markets.

Theorem 2 *If $\tilde{N}(\alpha, \beta, \gamma)$ is well defined, then*

$$\frac{\partial \tilde{N}(\alpha, \beta, \gamma)}{\partial \beta} > 0, \frac{\partial \tilde{N}(\alpha, \beta, \gamma)}{\partial \alpha} < 0, \frac{\partial \tilde{N}(\alpha, \beta, \gamma)}{\partial \gamma} > 0. \tag{14}$$

Proof: See online Appendix.■

We therefore find that the *upper bound to the equilibrium number of firms* in the industry is decreasing in the level of the intra-firm network effect and increasing in the level of inter-firm network spillovers and the overall network effect.

3 Conclusion

As in Sutton's (1998) model the upper bound on the equilibrium number of firms increases in the *overall network elasticity parameter*. However the reasoning underlying the effect is more complex. It follows from the fact that the lower bound to concentration will exist only if intra-firm network effects dominate inter-firm network spillovers and this *asymmetry* has to be stronger the weaker the overall network effect. If this asymmetry is sufficiently pronounced, a higher elasticity will unambiguously decrease individual quantities, increase profits and lead to more firms entering hence pushing the lower bound to concentration downwards. Larger intra-firm network effects (or lower inter-firm network spillovers) will lead to a decrease in the upper bound to the level of entering firms and hence move the lower bound to concentration upwards in large markets.

Besides the implications of network spillovers for large market the analysis also hints at the way that competition for market share will be different when goods are "new" and not well known yet. The threat of helping a competitor by selling more of its own products making *any variety* of the product more popular will effectively decrease the "overproduction" effect implied by having intra-firm network effects and thus makes entry into this industry more attractive than otherwise. Hence taking network spillovers into account puts the competitive concerns that followed the video game industry from its very beginning in a different perspective.

4 References

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5 Online Appendix

Proof of Proposition 1:

Derivation of equilibrium prices (6) and Cournot-Nash profits (7) in a differentiated product industry:

The profit of firm i with common marginal cost level c is given as

$$\pi_i = \lambda u_i x_i - c x_i \tag{15}$$

Using the finding above that goods with positive sales in equilibrium must have prices proportional to their qualities (3) we find λ from the total budget condition as

$$\lambda = \frac{S}{\left(\sum_j u_j x_j\right)} \quad j = 1, ...i, ...N.$$

The optimal quantity choice for firm i is found from its first order condition, i.e. by differentiating (15) with respect to quantity and setting it equal to zero

$$\frac{\partial \pi_i}{\partial x_i} = \lambda u_i + u_i x_i \frac{\partial \lambda}{\partial x_i} - c = 0 \tag{16}$$

Differentiation of λ w.r.t. some x_i yields

$$\frac{\partial \lambda}{\partial x_i} = -\frac{S}{\left(\sum_j u_j x_j\right)^2} \frac{\partial}{\partial x_i} \left(\sum_j u_j x_j \right) = -\frac{S u_i}{\left(\sum_j u_j x_j\right)^2} = -\frac{u_i}{S} \lambda^2$$

Substituting for $\frac{\partial \lambda}{\partial x_i}$ and rearranging we find optimal quantities for firm i to be

$$u_i x_i = \frac{S}{\lambda} - \frac{cS}{\lambda^2} \frac{1}{u_i} \tag{17}$$

and summation over all j firm's products yields

$$\sum_j u_j x_j = \frac{NS}{\lambda} - \frac{cS}{\lambda^2} \sum_j \frac{1}{u_j} \tag{18}$$

Using the total budget condition (4) again we can rewrite this as

$$\frac{S}{\lambda} = \frac{NS}{\lambda} - \frac{cS}{\lambda^2} \sum_j \frac{1}{u_j} \tag{19}$$

and find

$$\lambda = \frac{c}{N-1} \sum_j \frac{1}{u_j} \tag{20}$$

Substitution for λ into (17) yields optimal qualities as

$$x_i^* = \frac{S}{c} \frac{N-1}{u_i \sum_j \frac{1}{u_j}} \left\{ 1 - \frac{N-1}{u_i \sum_j \frac{1}{u_j}} \right\} \tag{21}$$

We now *solve for equilibrium prices* using $p_j = \lambda u_j \ \forall \ j = 1, ...i, ...N$ so that from (20) we find the price for good i as

$$p_i^* = \frac{cu_i}{N-1} \sum_j \frac{1}{u_j} \tag{22}$$

Alternatively we can write

$$p_i^* - c = \left\{ \frac{u_i}{N-1} \sum_j \frac{1}{u_j} - 1 \right\} c \tag{23}$$

Substituting (21) and (23) into the profit function (15) after some simplification we find equilibrium profits of

$$\pi_i^* = \left\{ 1 - \frac{N-1}{u_i \sum_{j=1}^{N} \frac{1}{u_j}} \right\}^2 S$$

which is precisely $\pi(u_i \,|\mathbf{u}_{-i})$ as given in (7) if we 'normalize' profits by defining $\pi(\cdot\,|\cdot) \equiv \frac{\pi_i^*}{S}$. ■

Proof of Proposition 2:

Total profits are given by the sum of stage two and stage three profits as

$$\Pi_i = (p-c)x_i + S\pi(u_i \,|\mathbf{u}_{-i}) \tag{24}$$

which due to the technology (8) depends on stage two quantities only.

Quality symmetry $\bar{\mathbf{u}} = \mathbf{i}$ in stage two implies that all products sell at a common price which we find from (4) as $p = \frac{S}{\mathbf{i}'\mathbf{x}}$ where column vector $\mathbf{i}$ contains a column of $1's$ and $\mathbf{x}$ denotes the total quantity vector.

The optimal stage two *quantity* is derived from the first order condition

$$\frac{\partial \Pi_i}{\partial x_i} - \left[\frac{S}{\mathbf{i}'\mathbf{x}} - \frac{S}{\mathbf{i}'\mathbf{x}\mathbf{x}'\mathbf{i}}x_i - c\right] = S\frac{\partial \pi}{\partial u_i}\frac{\partial u_i}{\partial x_i} + S[\nabla\pi(\mathbf{u}_{-i})]'\nabla\mathbf{u}_{-i}(x_i) = 0 \quad (25)$$

where the last term results from the fact that there will be spillovers from firm $i's$ quantity decision in stage two on the other firms' qualities in stage three. These affect firm $i's$ profits adversely. Hence there are *strategic effects* that firm i will take into account when deciding about output in stage two. The assumption of output symmetry at this stage implies that on the equilibrium path firms will also have symmetric qualities in stage three of the game, i.e. $(u_i) = \bar{u} \;\forall\; i$.

The first element on the RHS of the first order condition is

$$\frac{\partial \pi}{\partial u_i}\frac{\partial u_i}{\partial x_i}\Big|_{(u_i)=\bar{u}} = 2\frac{(N-1)^2}{N} \times \frac{\alpha}{\gamma}\frac{1}{\alpha + \mathbf{i}'\boldsymbol{\beta}}\frac{\pi}{x} \quad (26)$$

the second will be

$$[\nabla\pi(\mathbf{u}_{-i})]'\nabla\mathbf{u}_{-i}(x_i)\Big|_{(u_i)=\bar{u}} = -2\frac{(N-1)^2}{N} \times \frac{\mathbf{i}'\boldsymbol{\beta}}{\gamma(N-1)}\frac{1}{\alpha + \mathbf{i}'\boldsymbol{\beta}}\frac{\pi}{x} \quad (27)$$

derivations of both equations are below.

Inserting (26) and (27) and replacing π by $S\pi(\bar{\mathbf{u}})$ in (25) we find

$$\begin{aligned}\frac{\partial \Pi_i}{\partial x_i} &= \left[\frac{S}{\mathbf{i}'\mathbf{x}}(1 - \frac{x_i}{\mathbf{x}'\mathbf{i}}) - c\right] + 2\frac{(N-1)^2}{N} \times \frac{\alpha}{\gamma}\frac{1}{\alpha + \mathbf{i}'\boldsymbol{\beta}}\frac{S}{N^2 x} \\ &\quad -2\frac{(N-1)^2}{N} \times \frac{\mathbf{i}'\boldsymbol{\beta}}{\gamma(N-1)}\frac{1}{\alpha + \mathbf{i}'\boldsymbol{\beta}}\frac{S}{N^2 x} = 0 \end{aligned} \quad (28)$$

so that the own quantity effect on the average quality of the opponents is $\frac{\mathbf{i}'\boldsymbol{\beta}}{\alpha(N-1)}$ times the effect on the own quality level. In case of full *inter-firm spillover symmetry* ($\beta_i = \beta \;\forall\; i$) the first order condition simplifies to

$$\frac{\partial \Pi_i}{\partial x_i} = \frac{S}{Nx}(1 - \frac{1}{N}) - c + \frac{1}{\gamma}\frac{\alpha - \beta}{\alpha + \beta(N-1)}2\left[N + \frac{1}{N} - 2\right]\frac{S}{N^2 x} = 0 \quad (29)$$

which we can solve for the optimal per firm production in stage two as

$$x^* = \frac{S}{Nc}(1 - \frac{1}{N}) + \frac{1}{\gamma}\frac{\alpha - \beta}{\alpha + \beta(N-1)}2\left[N + \frac{1}{N} - 2\right]\frac{S}{N^2 c}$$

To satisfy the non-negativity constraint for output in case $\alpha < \beta$ we have that $\gamma \geq \frac{2}{N}$ (which is implied by the non-negativity condition on profits) so that $\gamma_{\alpha<\beta} \in [\frac{2}{N}, \infty)$ and $x^* \in [0, \frac{S}{c}\frac{N+1}{N^2}]$. As none of the following results requires that $\gamma \to 0$ the assumption does not constrain the generality of the analysis. ■

Intermediate steps in the Proof of Proposition 2:

Derivation of (26) and (27). Taking the derivative of the stage three profit function (9) with respect to u_i, the *quality* level of firm i, given all other firms have quality level $\bar{u}$, i.e. $(u_{-i}) = \bar{u}$ yields

$$\frac{\partial S\pi(u_i\,|\mathbf{u}_{-i})}{\partial u_i} = \frac{2S\bar{u}\left(u_i(N-1) + \bar{u}(2-N)\right)(N-1)^2}{\left(\bar{u} + u_i(N-1)\right)^3} \tag{30}$$

Write (30) as an elasticity of perceived quality on profits given that all firms have quality level $\bar{u}$, i.e. $(u_i) = \bar{u}\ \forall\ i$ of the form

$$\frac{u_i}{\pi}\frac{\partial \pi}{\partial u_i}\Big|_{(u_i)=\bar{u}} = 2\frac{(N-1)^2}{N} \tag{31}$$

Note that symmetrically we find that from the Cournot profit function (7)

$$\frac{u_j}{\pi}\frac{\partial \pi}{\partial u_j}\Big|_{(u_i)=\bar{u}} = -2\frac{(N-1)}{N} \tag{32}$$

so that firm $j's$ quality with $j \neq i$ will have a negative effect on firm $i's$ profit π so that elasticity of perceived quality u_j on profits π_i is negative. For a derivation see below.

Derivation of (31) and (32), the elasticity of perceived rival quality on firm $i's$ profits: From the Cournot profit function (7) *(note that the sum over j includes i)* with N firms we find that the derivative of profits for firm i with respect to its own quality level is

$$\frac{\partial \pi}{\partial u_i} = 2\left(1 - \frac{N-1}{u_i\sum_j \frac{1}{u_j}}\right)\left(\frac{1}{u_i^2(\sum_j \frac{1}{u_j})}\left(N - 1 - \frac{N-1}{u_i\sum_j \frac{1}{u_j}}\right)\right)$$

and the derivative of profits for firm i with respect to one other firm $j's$ quality level is

$$\frac{\partial \pi}{\partial u_j} = -2\left(1 - \frac{N-1}{u_i \sum_j \frac{1}{u_j}}\right)\left(\frac{N-1}{u_i \left(\sum_j \frac{1}{u_j}\right)^2 u_j^2}\right)$$

Writing these in terms of elasticities we find the desired results.

Rewriting *first part* of the right hand side (RHS) of the equation (25) as

$$\frac{\partial \pi}{\partial u_i}\frac{\partial u_i}{\partial x_i} = \left(\frac{u_i}{\pi}\frac{\partial \pi}{\partial u_i}\right) \times \left(\frac{\pi}{u_i}\frac{\partial u_i}{\partial x_i}\right) \tag{33}$$

using (31) and the elasticity formula for u_i which gives the elasticity of perceived quality in stage three with respect to stage two output as

$$\frac{x_i}{u_i}\frac{\partial u_i}{\partial x_i} = \frac{\alpha}{\gamma}\frac{x_i}{(\alpha x_i + \boldsymbol{\beta}'\mathbf{x}_{-i})} \tag{34}$$

we find

$$\frac{\partial \pi}{\partial u_i}\frac{\partial u_i}{\partial x_i}\Big|_{(u_i)=\bar{u}} = 2\frac{(N-1)^2}{N} \times \frac{\alpha}{\gamma}\frac{x_i}{(\alpha x_i + \boldsymbol{\beta}'\mathbf{x}_{-i})}\frac{\pi}{x_i} \tag{35}$$

The continuity of the support of α and β guarantees that $\alpha = (\beta_i) = 0$ $\forall$ i is a zero probability event. We seek a symmetric SPNE in which firms set a common output level x in stage two. We can therefore set $x_i = (\mathbf{x}_{-i}) = x$ to simplify the above to

$$\frac{\partial \pi}{\partial u_i}\frac{\partial u_i}{\partial x_i}\Big|_{(u_i)=\bar{u}} = 2\frac{(N-1)^2}{N} \times \frac{\alpha}{\gamma}\frac{1}{\alpha + \mathbf{i}'\boldsymbol{\beta}}\frac{\pi}{x} \tag{36}$$

We make an assumption about *full symmetry of spillovers* as follows. The complete network effect technology can be written as

$$\mathbf{u} = \max(1, (\mathbf{B}\mathbf{x}))^{\frac{1}{\gamma}} \tag{37}$$

where $\mathbf{B}$ is a fully symmetric $N \times N$ matrix with generic elements $(\beta_{ij}) = (\beta_{ji}) = \beta$ $\forall$ $i \neq j$ and $(\beta_{ij}) = \alpha$ $\forall$ $i = j$ on the diagonal and $\mathbf{u}$ and $\mathbf{x}$ are

$N \times 1$ column vectors of quality and quantity. This formulation can then be broken into one of the form

$$\mathbf{u} = \max(1, (\alpha \mathbf{x} + \boldsymbol{\Xi} \mathbf{x}))^{\frac{1}{\gamma}} \tag{38}$$

where $\boldsymbol{\Xi}$ is now a $N \times N$ matrix with $(\beta_{ij}) = 0 \; \forall \; i = j$ on the diagonal and β everywhere else. This guarantees that we can write the symmetric quality formula to (8) for some firm j as

$$u_j = \max(1, (\alpha x_j + \boldsymbol{\xi}_j' \mathbf{x}))^{\frac{1}{\gamma}} \tag{39}$$

where $\boldsymbol{\xi}_j'$ is the $j'th$ row of matrix $\boldsymbol{\Xi}$ of the generic form $(\beta_{j,1}, \beta_{j,2}, ...\beta_{j,j-1}, 0, \beta_{j,j+1}, ...\beta_{j,N})$. We now delete the empty diagonal by denoting $\boldsymbol{\beta}'_{-j}$ as the $1 \times N-1$ row vector without the $j'th$ element and $\mathbf{x}_{-j}$ as the $N-1 \times 1$ column vector of quantities without the $j'th$ element so that we can write

$$u_j = \max(1, (\alpha x_j + \boldsymbol{\beta}'_{-j} \mathbf{x}_{-j}))^{\frac{1}{\gamma}} \tag{40}$$

as the symmetric counterpart to (8) for firm j. Note that vector $\mathbf{x}_{-j}$ contains element (x_i).

Rewriting the *second part* of the RHS as

$$[\nabla \pi(\mathbf{u}_{-i})]' \nabla \mathbf{u}_{-i}(x_i) = \left(\frac{1}{\pi} [\nabla \pi(\mathbf{u}_{-i})]' \mathbf{u}_{-i} \right) \times \left([\nabla \mathbf{u}_{-i}(x_i)]' \frac{1}{\mathbf{u}_{-i}} \pi \right) \tag{41}$$

we find that the elasticity equation of quantity x_i with respect to the quality of an average other firm (under spillover symmetry 'any' other firm) $-i$ becomes

$$\nabla \mathbf{u}_{-i}(x_i) \frac{1}{\mathbf{u}'_{-i}} x_i = \frac{\mathbf{i}' \boldsymbol{\beta}}{\gamma(N-1)} \frac{x_i}{(\alpha x_i + \boldsymbol{\beta}' \mathbf{x}_{-i})} \tag{42}$$

so that using (32) for the aggregate of all $(N-1)$ other firm we find that

$$[\nabla \pi(\mathbf{u}_{-i})]' \nabla \mathbf{u}_{-i}(x_i) \left|_{(u_i)=\bar{u}} \right. = -2 \frac{(N-1)^2}{N} \times \frac{\mathbf{i}' \boldsymbol{\beta}}{\gamma(N-1)} \frac{x_i}{(\alpha x_i + \boldsymbol{\beta}' \mathbf{x}_{-i})} \frac{\pi}{x_i} \tag{43}$$

In a symmetric SPNE the above simplifies to

$$[\nabla \pi(\mathbf{u}_{-i})]' \nabla \mathbf{u}_{-i}(x_i) \left|_{(u_i)=\bar{u}} \right. = -2 \frac{(N-1)^2}{N} \times \frac{\mathbf{i}' \boldsymbol{\beta}}{\gamma(N-1)} \frac{1}{\alpha + \mathbf{i}' \boldsymbol{\beta}} \frac{\pi}{x} \tag{44}$$

■

Proof of Lemma 1:

See that the optimal stage two quantity x^* falls strictly in the level of spillovers β for all $N > 1$ as

$$\frac{\partial x^*}{\partial \beta} = -2\left[N + \frac{1}{N} - 2\right]\frac{S}{N}\frac{\alpha}{\gamma\left(\alpha + \beta(N-1)\right)^2 c} < 0 \tag{45}$$

and increases in the level of intra-firm network effect α as

$$\frac{\partial x^*}{\partial \alpha} = 2\left[N + \frac{1}{N} - 2\right]\frac{S}{N}\frac{\beta}{\gamma\left(\alpha + \beta(N-1)\right)^2 c} > 0 \tag{46}$$

It also increases if the overall network effect becomes more important, as

$$\frac{\partial x^*}{\partial \gamma} = -2\left[N + \frac{1}{N} - 2\right]\frac{S}{N^2}\frac{\alpha - \beta}{\gamma^2\left(\alpha + \beta(N-1)\right) c} < 0 \tag{47}$$

given that $\alpha > \beta$ holds and has the opposite sign if $\alpha < \beta$. ■

Proof of Proposition 3:

Given (10) stage two profits are

$$\begin{aligned}(p-c)x^* = \;& (\frac{S}{\mathbf{i}'\mathbf{x}} - c)x^* = \frac{S}{N} - cx^* = \\ & \frac{S}{N} - \left[\frac{S}{N}(1 - \frac{1}{N}) + \frac{1}{\gamma}\frac{\alpha - \beta}{\alpha + \beta(N-1)}2\left[N + \frac{1}{N} - 2\right]\frac{S}{N^2}\right] = \\ & \frac{S}{N^2} - \frac{1}{\gamma}\frac{\alpha - \beta}{\alpha + \beta(N-1)}2\left[N + \frac{1}{N} - 2\right]\frac{S}{N^2}\end{aligned} \tag{48}$$

and total profits are (given each firm earns profits $S\pi(\bar{\mathbf{u}}) = \frac{S}{N^2}$ in stage three)

$$\Pi_i = 2\frac{S}{N^2}\left\{1 - \frac{1}{\gamma}\frac{\alpha - \beta}{\alpha + \beta(N-1)}\left[N + \frac{1}{N} - 2\right]\right\} \tag{49}$$

To satisfy the non-negativity constraint for total profit in case $\alpha > \beta$ we assume that $\gamma \geq \frac{(N-1)^2}{N}$ so that $\gamma_{\alpha>\beta} \in [\frac{(N-1)^2}{N}, \infty)$ and $\Pi_i \in [0, \frac{S}{N} + \frac{S}{N^2}]$. ■

Proof of Proposition 4:

Entry will take place until the next firm will reduce profits below the sunk entry cost of $\varepsilon = 1$.[3] Hence the equilibrium number of firms is the largest integer N that satisfies

$$2\frac{S}{N^2}\left\{1-\frac{1}{\gamma}\frac{\alpha-\beta}{\alpha+\beta(N-1)}\left[N+\frac{1}{N}-2\right]\right\}\geq 1 \tag{50}$$

rearranging this equation yields the result. ■

Proof of Proposition 4 for a *continuous* number of firms:
The slope of LHS (13) which is now *binding* is

$$\frac{\partial(LHS)}{\partial N} = -2\left(\alpha-\beta\right)\left(N-1\right)\frac{N(\alpha+\beta)+\alpha-\beta}{\gamma\left(\alpha+\beta(N-1)\right)^2 N^2} \tag{51}$$

Thus LHS always attains an extreme point at $N = 1$ where it takes the value 2. Given $N = 1+\varepsilon$ it follows from (12) that $S > \frac{1}{2}$ so that the RHS takes the value $\frac{1}{S} < 2$, i.e. *below* the LHS. For $\alpha > \beta$ the LHS is strictly decreasing and will cut the strictly increasing parabola on the RHS at some finite equilibrium value N^* for any bounded value of S. Both functions are continuous so that *existence* of N^* follows from the intermediate value theorem and uniqueness from the strictness property of both sides.

For $\alpha \leq \beta$ and $N > 1$ the LHS is weakly increasing. The second derivative of the LHS is

$$\frac{\partial^2(LHS)}{\partial N^2} = 4\left(\alpha-\beta\right)\frac{(N^3-3N)(\alpha\beta+\beta^2)+2\beta^2+2\alpha\beta-\alpha^2}{\gamma\left(\alpha+\beta N-\beta\right)^3 N^3} \tag{52}$$

Again *existence* follows from the continuity of the functions. To check uniqueness for the case $\alpha \leq \beta$ where both the LHS and the RHS slope upwards we see from the above equation that the LHS is always weakly *concave* i.e. $\frac{\partial^2(LHS)}{\partial N^2} \leq 0$ for $N > 1$. As the RHS parabola is always strictly convex this guarantees a unique intersection of the two curves and hence a *unique* equilibrium number of firms. ■

Proof of Theorem 1:

[3] Note that having a sunk entry cost of ε (for example the initial cost to acquire a plant) is innocous here even though we will be talking about a bound for the equilibrium number of firms for the industry, since we will be looking at limit results and thus we can make ε arbitrary small relative to total industry demand.

An intercept with the $N - axis$ (and thus an upper bound on the equilibrium number of firms, $\tilde{N}(\alpha,\beta,\gamma)$) will exist if the LHS (13) *converges* to some negative number in the limit as $N \to \infty$ (not N^*). The limit result is

$$\lim_{N\to\infty} 2\left\{1 - \frac{1}{\gamma}\frac{\alpha-\beta}{\alpha+\beta(N-1)}\left[N + \frac{1}{N} - 2\right]\right\} = 2\frac{\beta(\gamma+1)-\alpha}{\gamma\beta} \tag{53}$$

Hence for $\alpha \leq \beta(\gamma+1)$ the LHS will converge to some non-negative number and no intercept exists. For for $\alpha > \beta(\gamma+1)$ such an intercept exists. For this not to violate the non-negativity constraint on total profits we require that $\gamma_{\alpha>\beta} \geq \frac{(N-1)^2}{N}$. ■

Proof of Theorem 2:

The two roots that satisfy the binding LHS (13) for $\alpha > \beta(\gamma+1)$ are

$$\tilde{N}_{1,2}(\alpha,\beta,\gamma) = \frac{1}{2}\frac{(\alpha-\beta)(\gamma+2) \pm \sqrt{\gamma}\sqrt{((\alpha-\beta)(\gamma(\alpha-\beta)+4\alpha))}}{\alpha-\beta(1+\gamma)} \tag{54}$$

Note that both roots will always be positive as $\alpha > \beta(\gamma+1) \Rightarrow \alpha > \beta$ so that the numerator and the denominator are always positive.$\tilde{N}_1(\alpha,\beta,\gamma)$ however does not satisfy $N > 1$ for $\beta \in [0,1]$ and $\alpha > \beta(\gamma+1)$. By contradiction:

$$\tilde{N}_1(\alpha,\beta,\gamma) > 1 \Leftrightarrow$$

$$\underbrace{\alpha - \beta - \frac{\sqrt{\gamma}}{\gamma}\sqrt{((\alpha-\beta)(\gamma(\alpha-\beta)+4\alpha))}}_{\Psi} > -2\beta \tag{55}$$

For $\alpha = \beta(\gamma+1)$ the above equation (55) holds with equality. We now show that Ψ is *strictly decreasing* in α. The derivative of Ψ w.r.t α is

$$\frac{\partial\Psi}{\partial\alpha} = 1 - \frac{\gamma(\alpha-\beta)+4\alpha+(\alpha-\beta)(\gamma+4)}{2\sqrt{\gamma}\sqrt{((\alpha-\beta)(\gamma(\alpha-\beta)+4\alpha))}} \tag{56}$$

thus we need to show that

$$\frac{\alpha(\gamma+4)-\beta(\gamma+2)}{\gamma} > \sqrt{(\alpha-\beta)^2 + \frac{4\alpha(\alpha+\beta)}{\gamma}}$$

Completion of the square yields

$$\frac{\alpha(\gamma+4)-\beta(\gamma+2)}{\gamma}+\frac{2\alpha}{\gamma}>\sqrt{\left((\alpha-\beta)+\frac{2\alpha}{\gamma}\right)^2}$$

from which we find $2\alpha-\beta>0$. This always holds as the condition $\alpha > \beta(\gamma+1) \Rightarrow \alpha > \beta$. Hence for some $\alpha' > \alpha = \beta(\gamma+1)$ we find

$$\alpha'-\beta-\frac{\sqrt{\gamma}}{\gamma}\sqrt{((\alpha'-\beta)\left(\gamma(\alpha'-\beta)+4\alpha\right))}<-2\beta \tag{57}$$

which is a contradiction to (55). We conclude that $0 < \tilde{N}_1(\alpha,\beta,\gamma) < 1$.

The comparative statics of the relevant root $\tilde{N}_2(\alpha,\beta,\gamma) \equiv \tilde{N}(\alpha,\beta,\gamma)$ can be seen from the binding LHS (13) by differentiating. As $\tilde{N}(\alpha,\beta,\gamma)$ denotes the intercept of LHS (13) with the $N-axis$ given $\alpha > \beta(\gamma+1)$ and LHS is monotonous and non-increasing for $\alpha \geq \beta$, we know that $\tilde{N}_2(\alpha,\beta,\gamma)$ will change accordingly. ■